Graph Densification

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Which graphs have a dense approximation?
Graph densification

**Given:** Graph $G$

**Find:** Graph $H$ such that

- $H$ is denser than $G$
- $H$ is a cut/spectral approximation of $G$
But... Why?

• Natural question about graphs
  – Which graphs are intrinsically sparse?
• Inverse of *Graph Sparsification*
  – Lots of recent work [BK,ST,SS,BSS,FHHP,KMP,...]
• Connection to Dense Model Theorem
  [GT,RTTV,TTV,...]
• New algorithms?
  – Max-Cut has PTAS on dense graphs [FK,GGR,...]
• Leads to interesting characterization
Main conceptual message

Either:

*Densifier*

$G$ has non-trivial cut densifier

Or:

*Embedding*

$G$ admits a weak embedding into $\ell^1$

Example:
Expander graph does not embed into $\ell^1$ [LLR], has densifier (complete graph)

#biwinning
Graphs are allowed to have edge weights in $[0,1]$

**Density** = sum of edge weights

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**Def.** $H$ is a one sided $C$-multiplicative cut approximation of $G$ if for every $S$:

$$e_H(S, S^C) \leq C \cdot e_G(S, S^C)$$

Short: $C$-approximation

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Really, there are six natural notions here:

one/two-sided, additive/multiplicative, spectral/cut

Note: Spectral stronger than cut
Recap: Metrics

Cut metric:
\[ \delta_S(u, v) = \begin{cases} 1 & \text{if } |\{u, v\} \cap S| = 1 \\ 0 & \text{o.w.} \end{cases} \]

\[ \ell_1 \text{-metric:} \quad \rho = \sum_S \lambda_S \delta_S, \quad \lambda_S \geq 0 \]

Graph metric:
\[ d_G(u, v) = \text{shortest path between } u \text{ and } v \text{ in } G \]
Results

Theorem:
A graph $G$ has a $C$-approximation of density $\alpha n^2$ iff $G$ does not have an $(\alpha, C)$-humble embedding into $\ell_1$

Definition (humble):
\[
\ell_1\text{-metric } \rho \text{ s.t. } (1 - \alpha)n^2 \text{ pairs of vertices } (u, v) \text{ satisfy}
\]
\[
\rho(u, v) \geq C \cdot \mathbb{E}_{e \in G} \rho(e)
\]

Note: To rule out densifier, exhibit $(o(1), O(1))$-humble embeddings
Examples

**Theorem:**
Planar graphs have \( (O(1/n), O(1)) \)-humble embeddings (and hence no non-trivial densifiers).

**Theorem:**
For every \( m \in [O(n), n^2] \), there is a random geometric graph with \( m \) edges that does not have an \( O(1) \)-approximation with density \( \Omega(mn^{0.01}) \).
Results (spectral)

**Theorem:**

$G$ has one-sided $C$-multiplicative spectral approximation of density $\alpha n^2$ iff $G$ does not have an $(\alpha, C)$-humble embedding into $\ell_2^2$.

**Theorem (cut vs spectral):**

If $G$ has a one-sided $C$-multiplicative *cut* approximation of density $m$, then $G$ has a one-sided $1.01C^2$-multiplicative *spectral* approximation of density $\Omega(m)$.

**Remark:**

Can compute optimal spectral densifier efficiently via SDP (= approximately optimal cut densifier by Theorem)
(Non-)Results in additive case

• Embedding approach less fruitful

• Theorems:
  – Can compute optimal additive cut densifier efficiently (unlike in multiplicative case) via [AN]
  – Cycle does not have a densififer with 2n edges

• Remark: Dense Model Theorem [RTTV] classifies which graphs have additive cut densifiers with $\Omega(n^2)$ edges.
Some intuition
Suppose you have a bounded degree graph $G$ that came with a non-contractive embedding $\rho$ into $\ell^1$ with small average stretch.

You also have an $O(1)$-approximation $H$

Claim: $H$ can’t be very dense

Why?
Observation:
Few close pairs
Many far pairs

Goal:
Argue most edges in $H$
must be short!
Let’s bound 

$$\frac{\sum_{(u, v) \in E_H} d_G(u, v)}{|E_H|}$$

Average distance in $G$ of edges in $H$
Let's bound

$$\mathbb{E}_{(u,v) \in E_H} d_G(u, v) \leq \mathbb{E}_{(u,v) \in E_H} \rho(u, v)$$

(Average stretch)

Average distance in G of edges in H (non-contractive)
Let’s bound

\[ \mathbb{E}_{(u, v) \in E_H} d_G(u, v) \leq \mathbb{E}_{(u, v) \in E_H} \rho(u, v) \]

(non-contractive)

\[ = \mathbb{E}_{(u, v) \in E_H} \sum_S \lambda_S \delta_S(u, v) \]

(ell1)

Average distance in $G$ of edges in $H$
Let’s bound

\[ \mathbb{E}_{(u, v) \in E_H} d_G(u, v) \]

\[
\leq \mathbb{E}_{(u, v) \in E_H} \rho(u, v) \quad \text{(non-contractive)}
\]

\[
= \mathbb{E}_{(u, v) \in E_H} \sum_S \lambda_S \delta_S(u, v) \quad \text{(ell1)}
\]

\[
= \sum_S \lambda_S \mathbb{E}_{(u, v) \in E_H} \delta_S(u, v)
\]
Let’s bound

\[ \mathbb{E}_{(u, v) \in E_H} \] \[ d_G(u, v) \]

\[ \leq \mathbb{E}_{(u, v) \in E_H} \rho(u, v) \] (non-contractive)

\[ = \mathbb{E}_{(u, v) \in E_H} \sum_S \lambda_S \delta_S(u, v) \] (ell1)

\[ = \sum_S \lambda_S \mathbb{E}_{(u, v) \in E_H} \delta_S(u, v) \]

\[ = \sum_S \lambda_S e_H(S, S^c) \]
Let’s bound

\[ \mathbb{E}_{(u, v) \in E_H} d_G(u, v) \]

\[
\leq \mathbb{E}_{(u, v) \in E_H} \rho(u, v) \quad \text{(non-contractive)}
\]

\[
= \mathbb{E}_{(u, v) \in E_H} \sum_S \lambda_S \delta_S(u, v) \quad \text{(ell1)}
\]

\[
= \sum_S \lambda_S \mathbb{E}_{(u, v) \in E_H} \delta_S(u, v)
\]

\[
= \sum_S \lambda_S e_H(S, S^c)
\]

\[
\leq C \sum_S \lambda_S e_G(S, S^c) \quad \text{(C-approximation)}
\]
Let’s bound

$$\mathbb{E}_{(u, \nu) \in E_H} d_G(u, \nu)$$

\[ \leq \mathbb{E}_{(u, \nu) \in E_H} \rho(u, \nu) \quad \text{(non-contractive)} \]

\[ = \mathbb{E}_{(u, \nu) \in E_H} \sum_S \lambda_S \delta_S(u, \nu) \quad \text{(ell1)} \]

\[ = \sum_S \lambda_S \mathbb{E}_{(u, \nu) \in E_H} \delta_S(u, \nu) \]

\[ = \sum_S \lambda_S e_H(S, S^c) \]

\[ \leq C \sum_S \lambda_S e_G(S, S^c) \quad \text{(C-approximation)} \]

\[ = C \mathbb{E}_{(u, \nu) \in E_G} \rho(u, \nu) \quad \text{(average stretch)} \]
Tight characterization

• Uses **LP/SDP duality**

• LP:
  – Objective: Max sum of edge weights of H
  – Constraints: H is a cut approximation of G
  – Variables: Edge weights of H

• SDP similar

• Dual program gives rise to humble embedding

• Cut vs spectral connection: Translate dual certificate from SDP to LP
Open problems

• Lack of understanding in the additive case
• Connection between cut/spectral multiplicative densifier in the two-sided case?
• Connection between densifiers and small set expansion?
• Killer application of densifier/embedding dichotomy?
Thank you

I still don't have all the answers. I'm more interested in what I can do next than what I did last.